TORSION CLASSES FOR ALGEBRAS WITH RADICAL SQUARE ZERO

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Abstract. To study the set $\mathsf{tors}A$ of torsion classes of a finite dimensional k-algebra A with n simple modules, we apply the sign-decomposition of $\mathsf{tors}A$, into the subsets $\mathsf{tors}_\epsilon A$ for all $\epsilon \in \{\pm 1\}^n$. For an algebra with radical square zero, we show that $\mathsf{tors}_\epsilon A$ corresponds bijectively with the set of faithful torsion classes of a certain hereditary algebra $A^!_\epsilon$ with radical square zero.

1. Introduction

Let A be a finite dimensional basic algebra over a field k and $\mathsf{mod}A$ the category of finitely generated right A-modules. In representation theory of finite dimensional algebras, torsion classes of the module category has been studied extensively from various perspective, where a full subcategory of $\mathsf{mod}A$ is called *torsion class* if it is closed under factor modules, extensions and isomorphisms. In this paper, we study the set $\mathsf{tors}A$ of torsion classes of $\mathsf{mod}A$. The recent developed τ-tilting theory [1] provides an effective tool to study $\mathsf{tors}A$ with partial order given by inclusion relations, however we need to restrict our study to torsion classes which are functorially finite. In general, it is very hard to classify non-functorially finite torsion classes.

The first aim of this note is to introduce a new approach, which we call *sign-decomposition*, to the classification problem of all torsion classes. The second aim is to classify all torsion classes over algebras with radical square zero as an application of the sign-decomposition. They are one of the most fundamental classes of algebras and firstly studied by Gabriel [3]. We also discuss a connection with τ-tilting theory.

2. Sign-decomposition

Let A be a finite dimensional basic k-algebra with n simple modules $S(1), \dots, S(n)$. We denote by $P(i)$ be the projective cover of $S(i)$ for $i = 1, \dots, n$.

Definition 1. For each $\epsilon \in \{\pm 1\}^n$, let

$$\mathsf{tors}_\epsilon A := \{\mathcal{T} \in \mathsf{tors}A \mid S(i) \in \mathcal{T} \Leftrightarrow \epsilon(i) = 1\}.$$

Clearly, $\mathsf{tors}A$ is a disjoint union of these 2^n subsets.

We note that $\mathsf{tors}_\epsilon A$ forms an interval in $\mathsf{tors}A$ whose minimal (respectively, maximal) element is the smallest torsion class containing all $S(i)$ (respectively, $P(i)$) such that $\epsilon(i) = 1$. We begin with the following observation.

The detailed version of this paper will be submitted for publication elsewhere.

Proposition 2. *Let $A \to B$ a surjective morphism of k-algebras. Then the following diagram commutes:*

$$\begin{tikzcd} \mathsf{tors}A \arrow[rr, "\overline{(-)}"] \arrow[dr, "sign"'] & \circlearrowright & \mathsf{tors}B \arrow[dl, "sign"] \\ & \{\pm 1\}^n & \end{tikzcd}$$

where $\overline{(-)}$ maps $\mathsf{tors}A \ni \mathcal{T} \mapsto \overline{\mathcal{T}} := \mathcal{T} \cap \mathsf{mod}B \in \mathsf{tors}B$.

Here, the surjectivity of $\overline{(-)}$ is shown in [2]. It is interesting when the map $\overline{(-)}$ restricts to an isomorphism from $\mathsf{tors}_\epsilon A$ to $\mathsf{tors}_\epsilon B$ for a given $\epsilon \in \{\pm 1\}^n$.

Next, we focus on a special class of torsion classes said to be *functorially finite*, i.e. such torsion classes of the form $\mathcal{T} = \mathsf{Fac}M$ for some $M \in \mathsf{mod}A$. The following is a basic result in τ-tilting theory, where τ is the Auslander-Reiten translation of A (For the definition of support τ-tilting modules we refer to [1]).

Theorem 3. [1] *There is a bijection between*

(1) $\mathsf{f}\text{-}\mathsf{tors}A$ *the set of functorially finite torsion classes of* $\mathsf{mod}A$ *and*
(2) $\mathsf{s}\tau\text{-}\mathsf{tilt}A$ *the set of isomorphism classes of basic support τ-tilting A-modules.*

Remark 4. (1) Any tilting module is precisely a faithful support τ-tilting module. We denote by $\mathsf{tilt}A$ the set of isomorphism classes of basic tilting A-modules.
(2) Up to isomorphisms, any basic support τ-tilting A-module M uniquely gives rise to a *τ-tilting pair* (M, P) with P a projective A-module. In this note, we will identify M with a pair (M, P) above.

A natural question is that what happens if we restrict the bijection to each $\epsilon \in \{\pm 1\}^n$. There is a simple answer in terms of the g-vectors of modules.

Definition 5. [1] Let $M \in \mathsf{mod}A$ and $P^{-1} \to P^0 \to M \to 0$ be a minimal projective presentation of M, where $P^0 = \bigoplus_{i=1}^n P(i)^{m_i}$ and $P^{-1} = \bigoplus_{i=1}^n P(i)^{m'_i}$. We define the *$g$-vector* $g_A^M \in \mathbb{Z}^n$ of M to be $g_A^M := (m_1 - m'_1, \ldots, m_n - m'_n)$. In addition, for a pair (M, P) of A-modules, let $g_A^{(M,P)} := g_A^M - g_A^P$.

We have the following.

Proposition 6. *For each $\epsilon \in \{\pm 1\}^n$, the map in Theorem 3 restricts to a bijection between*

(1) $\mathsf{f}\text{-}\mathsf{tors}_\epsilon A := \mathsf{tors}_\epsilon A \cap \mathsf{f}\text{-}\mathsf{tors}A$ *and*
(2) $\mathsf{s}\tau\text{-}\mathsf{tilt}_\epsilon A := \{(M, P) \in \mathsf{s}\tau\text{-}\mathsf{tilt}A \mid g_i^{(M,P)} \in \epsilon(i) \cdot \mathbb{Z}_{>0}, i = 1, \ldots, n\}$.

3. Algebras with radical square zero

In this section, we study torsion classes for algebras with *radical square zero*, namely, such algebras that the square of the Jacobson radical is zero. Let A be a k-algebra with radical square zero and having n simple modules. A main result is the following.

Theorem 7. *Let A be an algebra with radical square zero. For each $\epsilon \in \{\pm 1\}^n$, there is a hereditary algebra $A_\epsilon^!$ with radical square zero and there is an isomorphism of partially ordered sets between*

(1) $\mathsf{tors}_\epsilon A$ *and*
(2) $\mathsf{fa\text{-}tors} A^!_\epsilon$ *the set of torsion classes of* $\mathsf{mod} A^!_\epsilon$ *which are faithful, i.e. containing all injectives.*

which induces a bijection between

(1)' $\mathsf{s}\tau\text{-}\mathsf{tilt}_\epsilon A$ *and*
(2)' $\mathsf{tilt} A^!_\epsilon$.

In particular, we have bijections

$$\mathsf{tors} A \overset{1-1}{\longleftrightarrow} \bigsqcup_{\epsilon\in\{\pm 1\}^n} \mathsf{fa\text{-}tors} A^!_\epsilon \quad \textit{and} \quad \mathsf{s}\tau\text{-}\mathsf{tilt} A \overset{1-1}{\longleftrightarrow} \bigsqcup_{\epsilon\in\{\pm 1\}^n} \mathsf{tilt} A^!_\epsilon.$$

Remark 8. We can describe the hereditary algebra $A^!_\epsilon$ above in terms of valued quivers. Let Γ be a valued quiver of A. Then a valued quiver of $A^!_\epsilon$ coincides with the following bipartite quiver $\Gamma^!_\epsilon$:

- the set of vartices is the same as Γ.
- we draw an arrow $i \leftarrow j$ if there is a valued arrow $i \to j$ in Γ with $\epsilon(i) = 1$, $\epsilon(j) = -1$ and assigning to this arrow the same valuation.

In particular, if Γ is simply a quiver, then $A^!_\epsilon$ is precisely a path algebra of $\Gamma^!_\epsilon$.

The bijection in Theorem 7 is explicitly described in the level of g-vectors. Now, for a given $\epsilon \in \{\pm 1\}^n$, we denote by $T_{(M,P)}$ the tilting $A^!_\epsilon$-module corresponding to $(M,P) \in \mathsf{s}\tau\text{-}\mathsf{tilt}_\epsilon A$.

Theorem 9. *Let* $\epsilon \in \{\pm 1\}^n$ *and* $(M,P) \in \mathsf{s}\tau\text{-}\mathsf{tilt}_\epsilon A$. *The diagonal matrix* $\mathrm{B}_\epsilon := \mathrm{diag}(\epsilon(1),\ldots,\epsilon(n))$ *transforms the g-vector* $g_A^{(M,P)}$ *of* (M,P) *into the dimension vector* $c_{A^!_\epsilon}^{T_{(M,P)}}$ *of* $T_{(M,P)}$, *that is,* $g_A^{(M,P)} = \mathrm{B}_\epsilon \cdot c_{A^!_\epsilon}^{T_{(M,P)}}$ *holds.*

4. Example

In this example we consider a quiver Γ: $(1 \rightrightarrows\!\!\leftleftarrows 2)$ and $A := k\Gamma/I$ an algebra with radical square zero, where I is an ideal generated by all path of length 2. By Remark 8, we have $A^!_\epsilon = k\Gamma^!_\epsilon$ a path algebra of $\Gamma^!_\epsilon$ for each $\epsilon \in \{\pm 1\}^2$, where

$$\Gamma^!_{(1,1)}\colon (1 \quad 2)\,, \quad \Gamma^!_{(1,-1)}\colon (1 \leftleftarrows 2)\,, \quad \Gamma^!_{(-1,1)}\colon (1 \rightrightarrows 2)\,, \quad \Gamma^!_{(-1,-1)}\colon (1 \quad 2).$$

When $\epsilon = (1,1)$ or $(-1,-1)$, the corresponding algebra is semisimple, so there is a unique tilting module itself with dimension vector $(1,1) \in \mathbb{Z}^2$. On the other hand, there are infinitely many tilting modules over Kronecker algebras, and their dimension vectors are given by $(2a+1, 2a+3)$ of preprojectives and $(2a+3, 2a+1)$ of preinjectives for $a = 0, 1, \ldots$.

By Theorem 9, the set of g-vectors of support τ-tilting A-modules is given by

$$\{(1,1)\}\sqcup\{(-2a-1, 2a+3), (-2a-3, 2a+1)\}\sqcup\{(2a+1, -2a-3), (2a+3, -2a-1)\}\sqcup\{(-1,-1)\}$$

for $a = 0, 1, \ldots$. We note that there are non-functorially finite torsion classes of $\mathsf{mod} A$ on the dotted line in Figure 1, that come from Kronecker algebras by Theorem 7. They form a partially ordered set isomorphic to $2^{\mathbb{P}^1(k)}$ the power set of $\mathbb{P}^1(k)$.

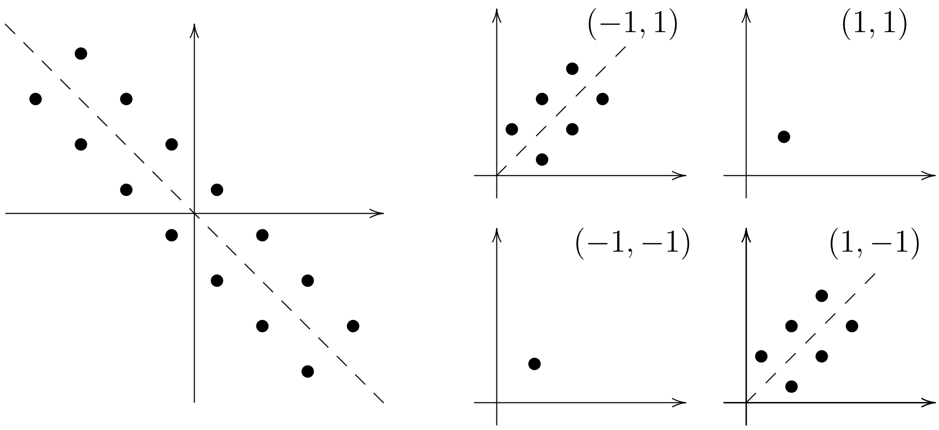

FIGURE 1. The set of g-vectors for $\mathsf{s}\tau\text{-}\mathsf{tilt}A$ in the left side and the dimension vectors for $\mathsf{tilt}A^!_\epsilon$ with $\epsilon \in \{\pm 1\}^2$ in the right side.

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